Uitwerking Tentamen Analyse 6 november 2012

1. (a) (a_n) convergeert niet omdat er twee deelrijen zijn die naar een verschillende waarde convergeren. **(2 pt.)**
   De limietpunten van A zijn 2 en -2 **(3 pt.)**, dus $\overline{A} = \{a_1, a_2, \cdots, -2, 2\}$ **(1 pt.)**.

   (b) F is de kleinste gesloten verzameling die E^c bevat, en dus is F^c de grootste open verzameling bevat in E. **(5 pt.)**
   Nee; $E = [0,1]$ **(2 pt.)**.

2. Define the continuous function $g(x) = f(x) - x$. Then $g(a) \leq 0$ and $g(b) \geq 0$. By the intermediate value theorem there exists a $c \in [a,b]$ such that $g(c) = 0$, or equivalently $f(c) = c$.

3. Consider $f_\epsilon(x_2) = f_\epsilon(x_1)$ with $x_1 \leq x_2$, or equivalently $\epsilon(g(x_2) - g(x_1)) = x_2 - x_1$. By the mean value theorem there exists $c \in (x_1, x_2)$ such that $\epsilon g'(c)(x_2 - x_1) = x_2 - x_1$ and hence

   $$|\epsilon||g'(c)||x_2 - x_1| = |x_2 - x_1|$$

   By taking ϵ such that $|\epsilon| < 1/M$ it follows that $x_1 = x_2$, and thus for these values of ϵ f_ϵ is one-to-one.

4. (a) For $x < 1$ we have $x^n \to 0$ and thus $f_n(x) \to 0$.
   For $x = 1$ we have $f_n(1) = \frac{1}{2}$ and thus $f(1) = \frac{1}{2}$.
   For $x > 1$ we have $x^n \to \infty$ and thus $f_n(x) \to 1$. **(3 pt.)**
   If f_n would converge uniformly then the limit function f would be continuous (since all f_n are continuous). Contradiction.**(3 pt.)**

   (b) For $x < 1$ $|\frac{x^n}{1+x^n} - 0| \leq x^n \leq c^n < \epsilon$ for $n > \frac{\ln|\epsilon|}{\ln c}$.**(4 pt.)**

   (c) For $x > 1$ $|\frac{x^n}{1+x^n} - 1| = \frac{1}{1+x^n} \leq \frac{1}{1+b^n} < \epsilon$ for n large enough.**(4 pt.)**
   Take the sequence $x_n := 1 + 1/n > 1$. Then $f_n(x_n) = \frac{(1+\frac{1}{n})^n}{!+(1+\frac{1}{n})^n} \to \frac{e}{1+e}$. Thus no uniform convergence to $f(x) = 1$ on $(1, \infty)$.**(5 pt.)**

5. (a) For $x < 1$ x^n converges quicker to 0 than n^2 to ∞. Furthermore $f_n(1) = 0$.**(2 pt.)**

   (b)
   $$\int f_n = n^2\left(\frac{1}{n+1}x^{n+1} - \frac{1}{n+2}x^{n+2}\right) = n^2\frac{1}{(n+1)(n+2)} \to 1$$
   (5 pt.)

   (c) If f_n would converge uniformly then $\int f_n \to \int f = 0$. (Or direct proof.)**(4 pt.)**

6. (a) $\frac{1}{\sin x + n^2} \leq \frac{1}{n^2-1}$, and thus by the Weierstrass test uniform convergence. Since the functions $\frac{1}{\sin x + n^2}$ are continuous this implies continuity of the sum function.**(5 pt.)**

   (b) $\frac{d}{dx}\left(\frac{1}{\sin x + n^2}\right) = \frac{-\cos x}{(\sin x + n^2)^2}$ and since $|\frac{-\cos x}{(\sin x + n^2)^2}| \leq \frac{1}{(n^2-1)^2}$ the series of derivatives converges uniformly, again by the Weierstrass test. Thus the series converges and f is differentiable. **(5 pt.)** Furthermore $f'(x) = \sum_{n=2}^{\infty} \frac{-\cos x}{(\sin x + n^2)^2}$, which is continuous (again by uniform convergence thanks to the Weierstrass test).**(4 pt.)**

7. Since f is continuous on $[a,b]$ the function $F(x) := \int_a^x f(t)dt$ is differentiable on $[a,b]$ with $F'(x) = f(x)$. By the mean value theorem there exists $c \in (a,b)$ such that $F(b) - F(a) = F'(c)(b-a)$, i.e. the required identity.(**9 pt.**) If f is bounded and continuous on (a,b) then the integral $\int_a^b f(t)dt$ still exists and $F(x)$ is continuous on $[a,b]$ and differentiable on (a,b); enough for the mean value theorem.(**2 pt.**)